Mathematics of Computation Meets Geometry

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November 3, 2018

Structure-preservation for ODEs:

Symplectic integration

Example: ODE initial value problem

Nonlinear pendulum with damping: $\ddot{\theta} = -\frac{g}{L}\sin\theta - \alpha\dot{\theta}$

Euler's method with 20,000 steps (1,000/sec for 20 sec)

method	L_h	max error
Euler	$O(h)$	$49°$
Leapfrog	$O(h^2)$	$0.24°$
Runge–Kutta	$O(h^4)$	$0.000048°$

A challenging problem: long-term stability of the solar system

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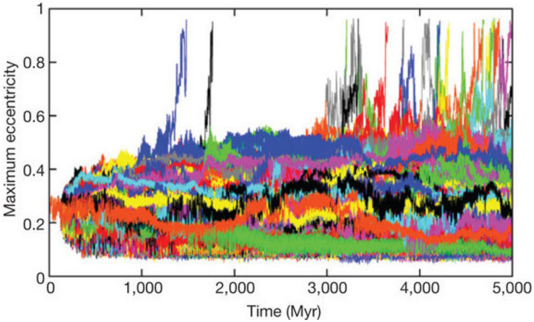

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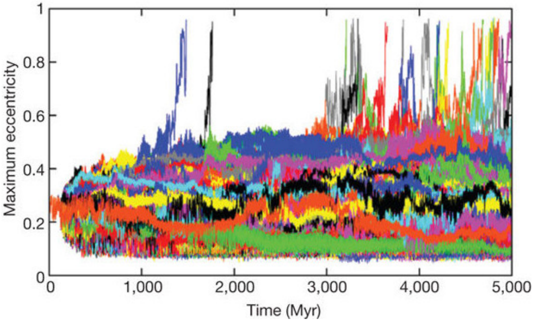
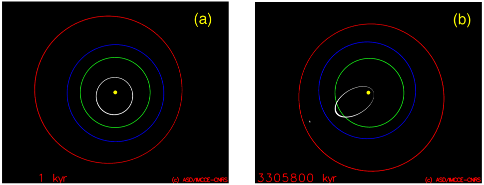
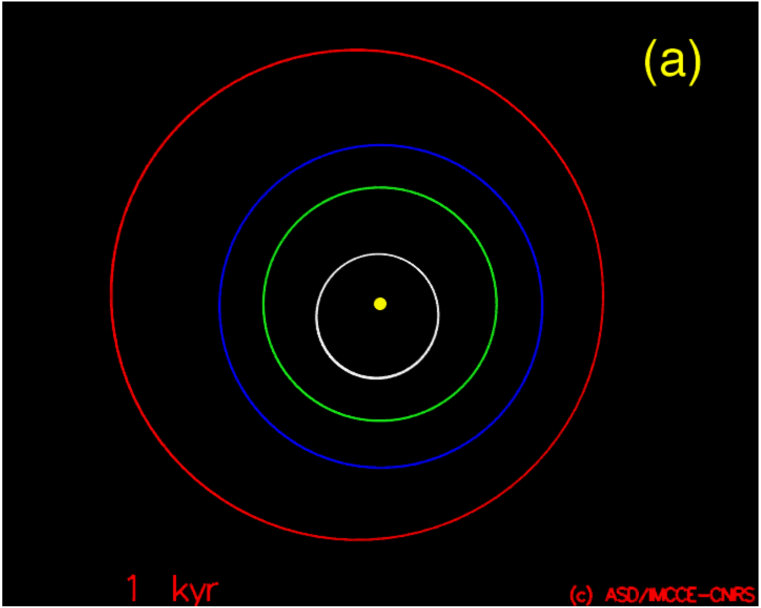


1% of the simulations resulted in unstable or collisional orbits.

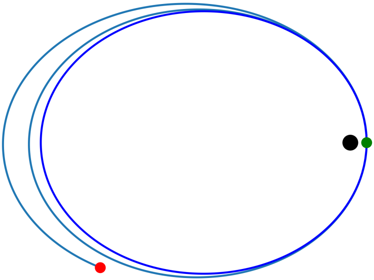




J. Laskar and M. Gastineau

Two 1st order methods for the Kepler problem

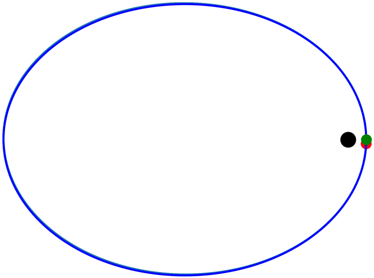
4 periods, 50,000 steps/period



Euler

$$\frac{x_{n+1} - x_n}{h} = v_n$$

$$\frac{v_{n+1} - v_n}{h} = -\frac{x_n}{|x_n|^3}$$



symplectic Euler

$$\frac{x_{n+1} - x_n}{h} = v_n$$

$$\frac{v_{n+1} - v_n}{h} = -\frac{x_{n+1}}{|x_{n+1}|^3}$$

Symplecticity and Hamiltonian systems

The (undamped) pendulum, Kepler problem, and the n-body problem are all Hamiltonian systems: they have the form

$$\dot{\boldsymbol{p}} = -\frac{\partial H}{\partial \boldsymbol{q}}, \quad \dot{\boldsymbol{q}} = \frac{\partial H}{\partial \boldsymbol{p}}, \qquad p, q : \mathbb{R} \to \mathbb{R}^d$$

This is a *geometric property:* it means that the flow map

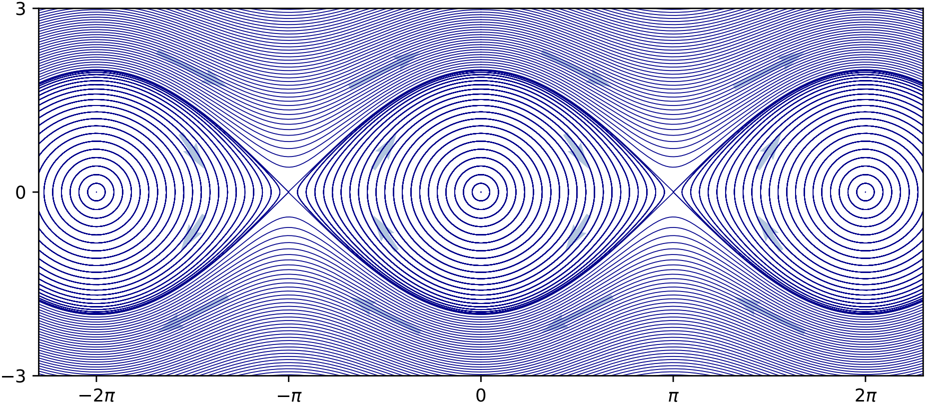
$$(\boldsymbol{p}_0, \boldsymbol{q}_0) \mapsto (\boldsymbol{p}(t), \boldsymbol{q}(t))$$

is a *symplectic transformation* for every t, i.e., the differential 2-form

$$dp^1 \wedge dq^1 + \cdots + dp^d \wedge dq^d$$

is invariant under pullback by the flow.

In 2D, $dp \wedge dq$ is the volume form so it is invariant $\iff$ the flow is volume-preserving.



Pendulum equations: $\dot{q} = p, \quad \dot{p} = -\sin q$

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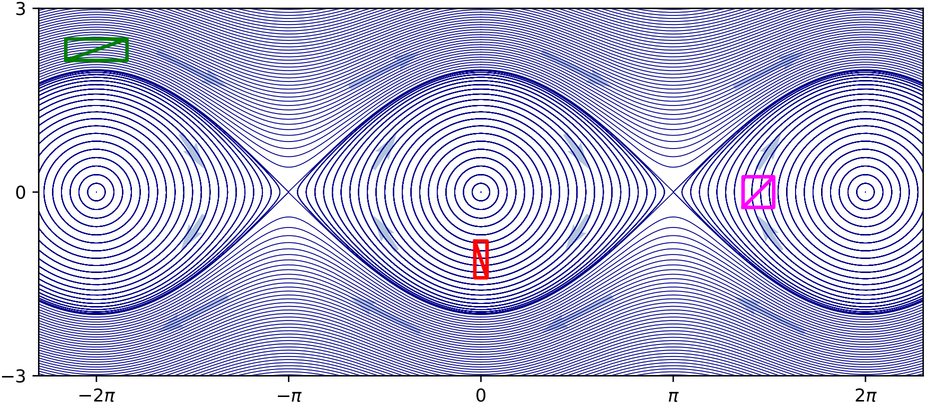

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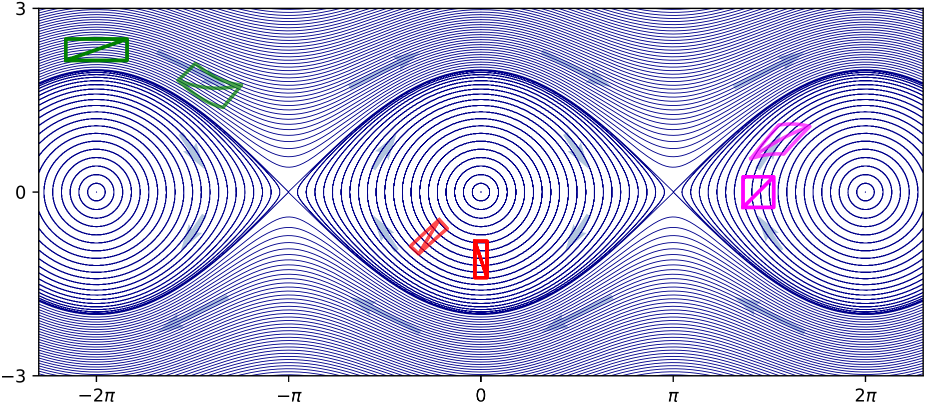

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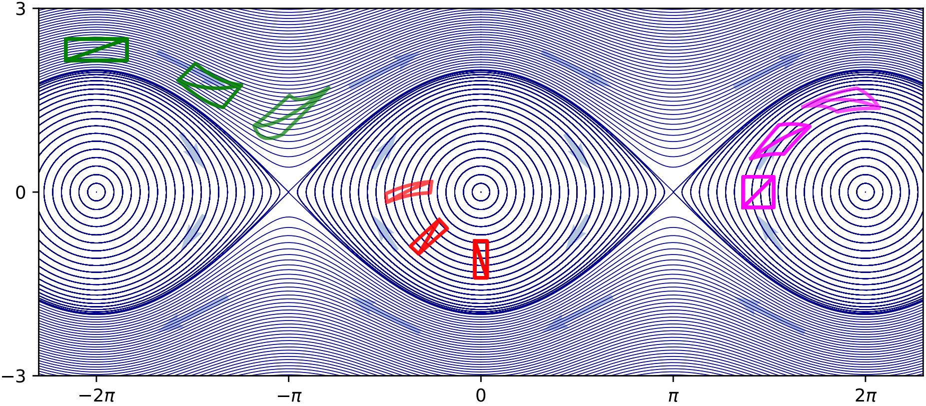

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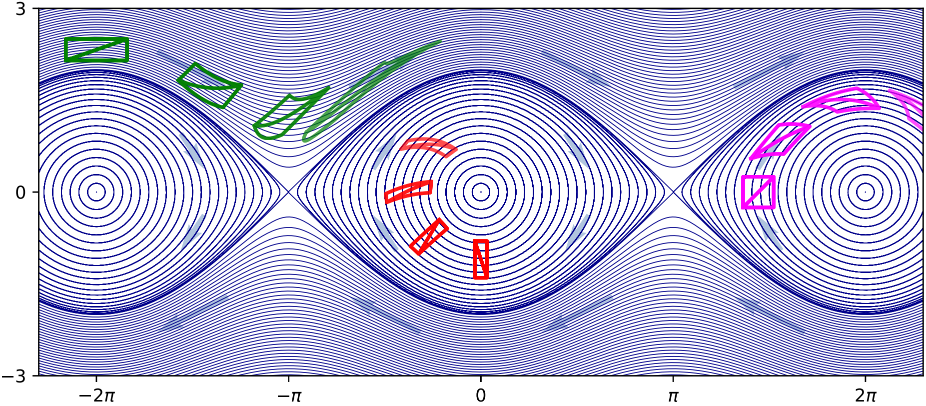

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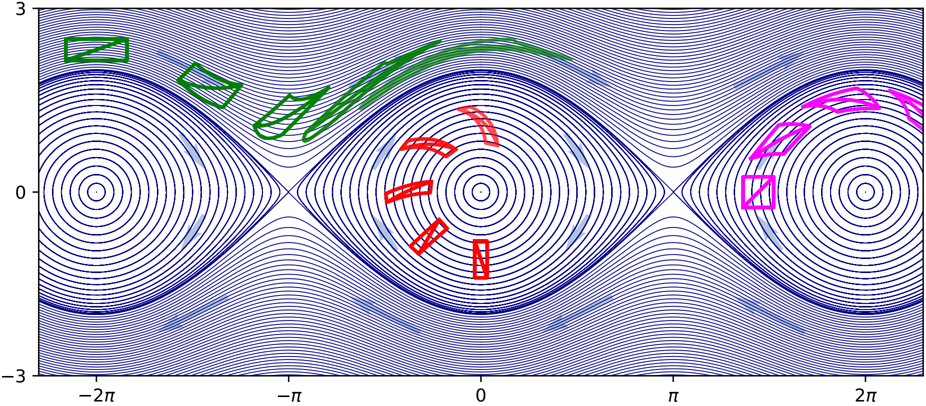

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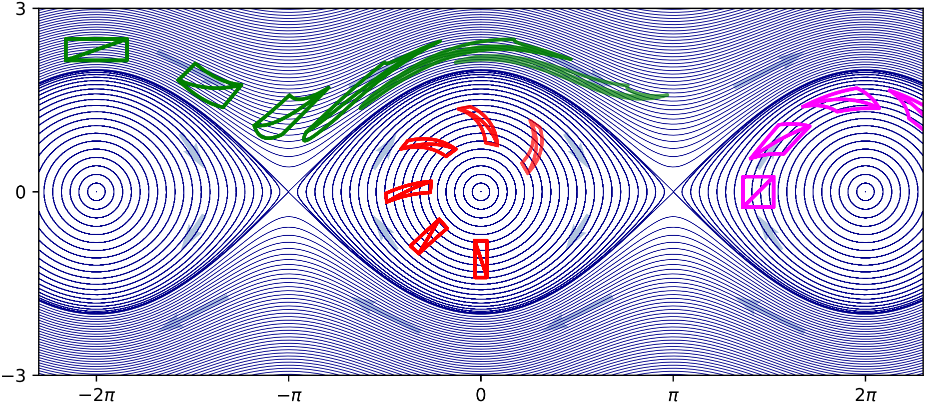

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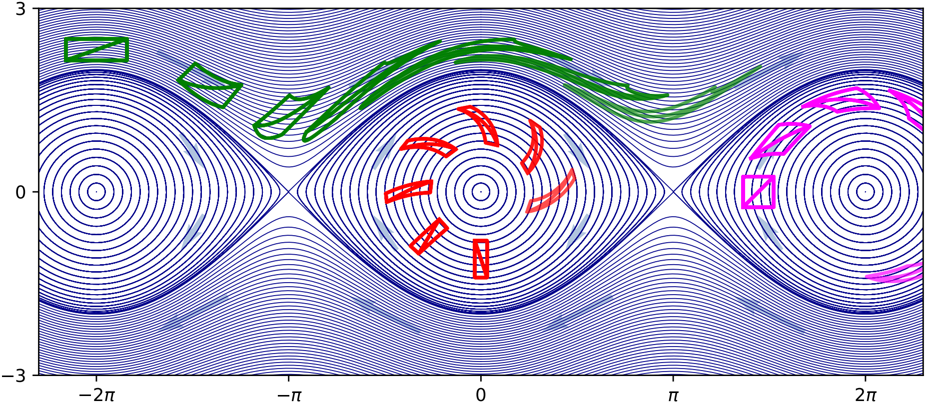

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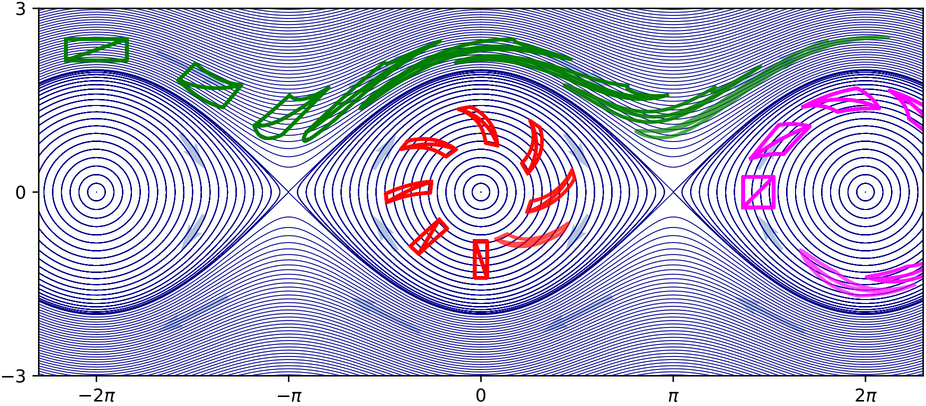

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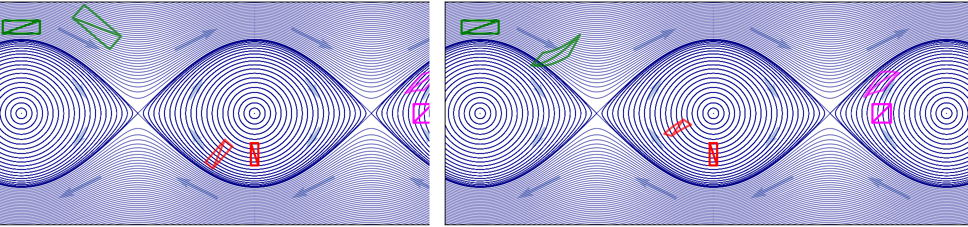
Symplectic discretization

Definition. *A discretization is symplectic* if the discrete flow map

$$(p_n, q_n) \mapsto (p_{n+1}, q_{n+1})$$

is a symplectic transformation (when the method is applied to Hamiltonian system).

The symplectic form must be preserved *exactly*, not to $O(h^r)$.



Euler   symplectic Euler

Sophisticated methods have been devised to find symplectic methods of high order, low cost, and with other desirable properties.

Backward Error Analysis

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Ordinary error analysis: How much do we change the true solution to obtain the discrete solution?

BEA: How much do we change the true problem to obtain the problem that the discrete solution solves exactly?

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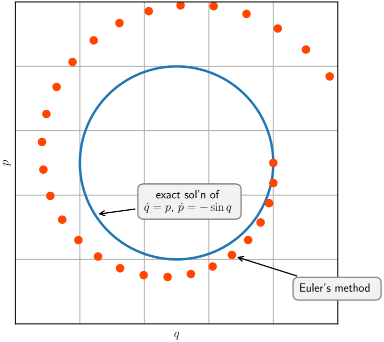

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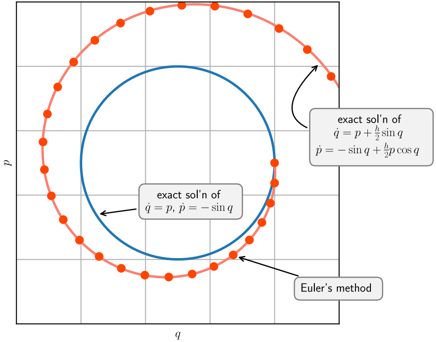

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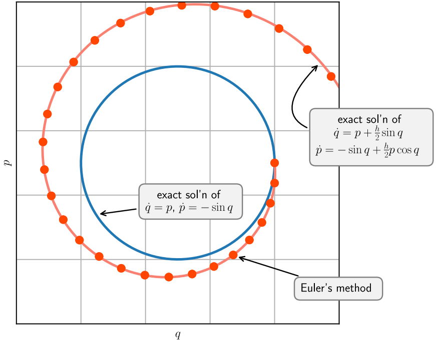

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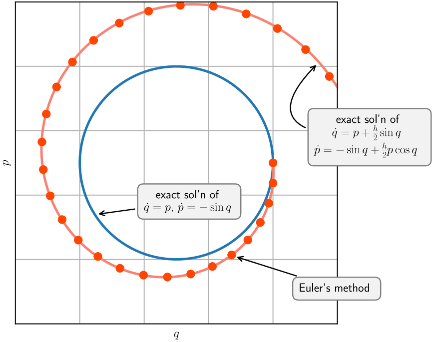
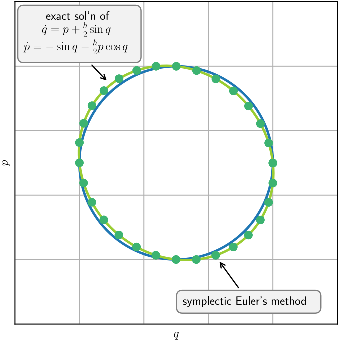
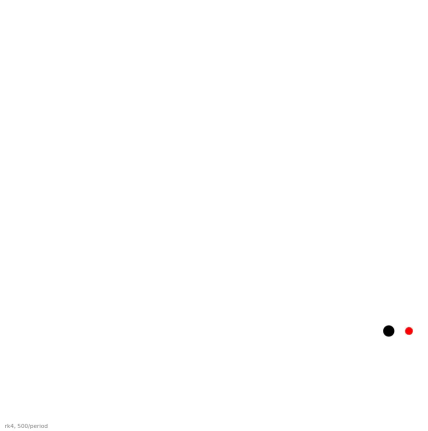
For a symplectic discretization, the modified equation is itself Hamiltonian. Therefore the discrete solution exhibits Hamiltonian dynamics: no dissipation, sources, sinks, spirals, …



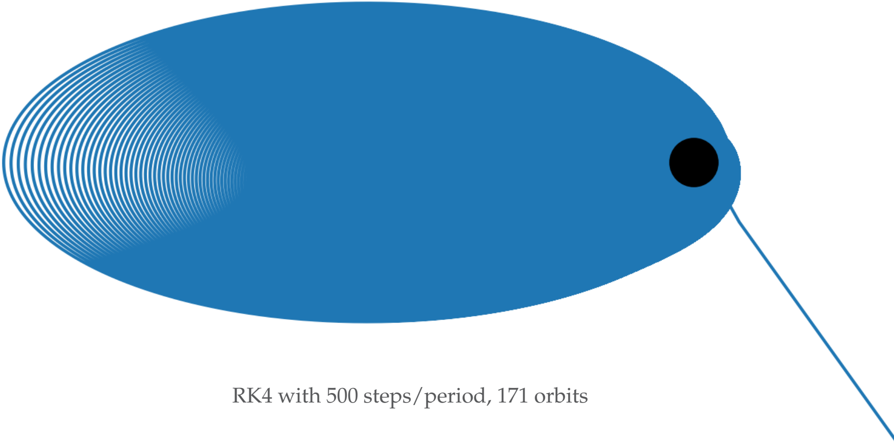


The Kepler problem using RK4



RK4 with 500 steps/period

Simplest planetary simulation: the Kepler problem using RK4



RK4 with 500 steps/period, 171 orbits

The Kepler problem using Calvo4

Calvo4 with 500 steps/period

Long-term simulation of the solar system

How did Laskar & Gastineau simulate the solar system for 5 Gyr?

They used SABA4, derived by McLachlan ’95, Laskar & Robutel ’00 using Lie theory and the Baker–Campbell–Hausdorff formula.

- symplectic
- preserves time-symmetry
- step length = 9 days, 200 billion steps
- 2nd order !?

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$$\text{consistency error} = O(\epsilon^2 h^2) + O(\epsilon h^8), \quad \epsilon = \frac{\text{planetary mass}}{\text{solar mass}} \approx 0.001$$

Some milestones

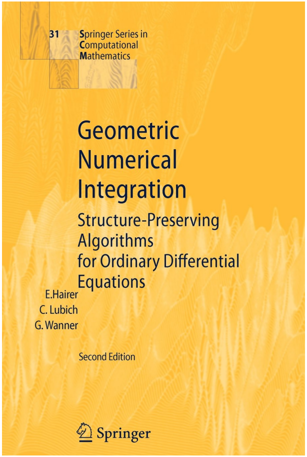
De Vogelaere 1956
Verlet 1967
Ruth 1983

Feng Kang 1985

Sanz-Serna and Calvo 1994

Reich 2000, Hairer–Lubich 2001

and many many more



Structure-preservation for PDEs:

Finite Element Exterior Calculus

Some milestones

1970s: golden age of mixed finite elements; Brezzi, Raviart–Thomas, Nédélec, ...

Bossavit 1988: *Whitney forms: a class of finite elements for 3D electromagnetism*

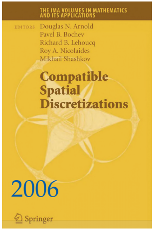
Hiptmair 1999: *Canonical construction of finite elements*

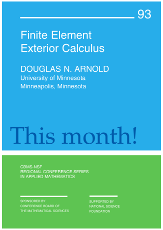
DNA @ ICM 2002: *Differential complexes and numerical stability*

DNA-Falk-Winther:

2006: *Finite element exterior calculus, homological techniques, and applications*

2010: *Finite element exterior calculus: from Hodge theory to numerical stability*





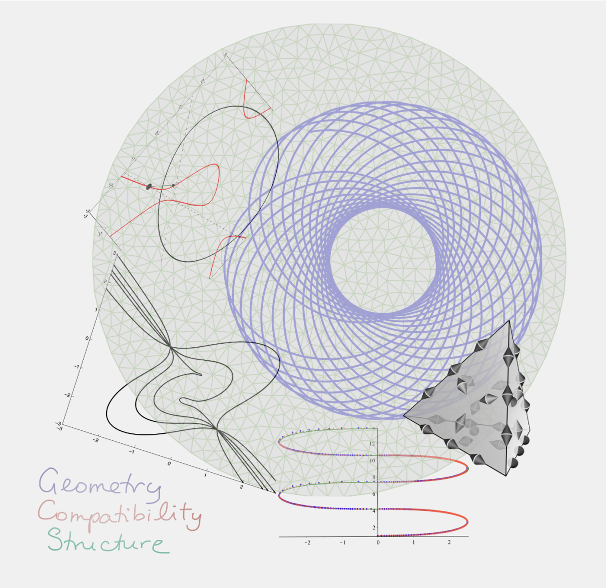
And many more: Awanou, Boffi, Buffa, Christiansen, Cotter, Demlow, Gillette, Gúzman, Hirani, Holst, Licht, Monk, Neilan, Rapetti, Schöberl, Stern, ...

Geometry, compatibility and structure preservation in computational differential equations

3 July 2019
to
19 December 2019

Isaac Newton Institute

Cambridge



De Rham complex

On a domain in 3D, the *L^2 de Rham complex* is

$$0 \to L^2 \xrightarrow{\text{grad}, H^1} L^2 \otimes \mathbb{R}^3 \xrightarrow{\text{curl}, H(\text{curl})} L^2 \otimes \mathbb{R}^3 \xrightarrow{\text{div}, H(\text{div})} L^2 \to 0$$

This is a special case of the L^2 de Rham complex on an arbitrary Riemannian n-manifold:

$$0 \to L^2\Lambda^0 \xrightarrow{d, H\Lambda^0} L^2\Lambda^1 \xrightarrow{d, H\Lambda^1} \cdots \xrightarrow{d, H\Lambda^{n-2}} L^2\Lambda^{n-1} \xrightarrow{d, H\Lambda^{n-1}} L^2\Lambda^n \to 0$$

Both may be seen as special cases of the structure of a *closed Hilbert complex*, a chain complex in the setting of unbounded operators in Hilbert space:

$$0 \to W^0 \xrightarrow{d, V^0} W^1 \xrightarrow{d, V^1} \cdots \xrightarrow{d, V^{n-1}} W^n \to 0$$

where each $d : W^i \to W^{i+1}$ is a *closed unbounded operator between Hilbert spaces with dense domain V^i and closed range* and $d \circ d = 0$.

AFW 2010, Brüning–Lesch 1992

The Hilbert complex structure

A closed Hilbert complex carries a lot of structure.

$$0 \to W^0 \underset{d^*,V_1^*}{\overset{d,V^0}{\rightleftarrows}} W^1 \underset{d^*,V_2^*}{\overset{d,V^1}{\rightleftarrows}} \cdots \underset{d^*,V_n^*}{\overset{d,V^{n-1}}{\rightleftarrows}} W^n \to 0$$

- Null space and range: $\mathfrak{Z}^k = \mathcal{N}(d^k)$ and $\mathfrak{B}^k := \mathcal{R}(d^{k-1})$ satisfy $\mathfrak{B}^k \subset \mathfrak{Z}^k$.
- Cohomology spaces: $H^k := \mathfrak{Z}^k / \mathfrak{B}^k$, key "geometric quantities". (For the de Rham complex their dimensions are the Betti numbers).
- Duality: Each d has an adjoint d^* leading to a dual Hilbert complex.
- Hodge Laplacian: $\Delta^k := dd^* + d^*d : W^k \to W^k$.
- Harmonic forms: $\mathfrak{H}^k = \mathfrak{Z}^k \cap \mathfrak{Z}_k^*$ realizes the cohomology space inside W^k. It is the null space of Δ^k.
- Hodge decomposition: $W^1 = \underbrace{\overbrace{\mathfrak{B}^k \oplus \mathfrak{H}}^{\mathfrak{Z}}}_{} \oplus \overbrace{\mathfrak{B}_k^*}^{\mathfrak{Z}^\perp}$, with $\underbrace{\mathfrak{B}^k}_{\mathfrak{Z}^{*\perp}}$ and $\underbrace{\mathfrak{H} \oplus \mathfrak{B}_k^*}_{\mathfrak{Z}^*}$
- Poincaré inequality: $\|u\| \leq C_P \|du\|, \quad u \in V^k \cap \mathfrak{Z}^\perp$

Hodge Laplacian

Whenever we have a segment $W^{k-1} \xrightarrow{d,V^{k-1}} W^k \xrightarrow{d,V^k} W^{k+1}$ of a Hilbert complex, we may consider the Hodge Laplace problem $\Delta^k u = f$. It has a solution iff $f \perp \mathfrak{H}^k$. The solution is unique up to an element of $\mathfrak{H}^k$.

- *Primal weak form:* Find $u \in V^k \cap V_k^* \cap \mathfrak{H}^\perp$ s.t.

$$\langle du, dv\rangle + \langle d^*u, d^*v\rangle = \langle f, v\rangle, \quad v \in V^k \cap V_k^* \cap \mathfrak{H}^\perp$$

- *Mixed weak form:* Find $\sigma \in V^{k-1}, u \in V^k, p \in \mathfrak{H}$ s.t.

$$\begin{aligned} \langle \sigma, \tau\rangle - \langle u, d\tau\rangle &= 0, && \tau \in V^{k-1}, \\ \langle d\sigma, v\rangle + \langle du, dv\rangle + \langle p, v\rangle &= \langle f, v\rangle, && v \in V^k, \\ \langle u, q\rangle &= 0, && q \in \mathfrak{H}. \end{aligned}$$

The two formulations are completely equivalent and both are well-posed (Hodge decomposition and Poincaré inequality).

The de Rham complex in 3D

$$0 \to L^2 \xrightarrow{\text{grad}, H^1} L^2 \otimes \mathbb{R}^3 \xrightarrow{\text{curl}, H(\text{curl})} L^2 \otimes \mathbb{R}^3 \xrightarrow{\text{div}, H(\text{div})} L^2 \to 0$$

$k = 0$: $\quad 0 \longrightarrow L^2(\Omega) \xrightarrow{(\text{grad}, H^1)} L^2(\Omega) \otimes \mathbb{R}^3$

Mixed=Primal: $\quad u \in H^1, p \in \mathbb{R}$: $\langle \text{grad}\, u, \text{grad}\, v \rangle = \langle f - p, v \rangle, v \in H^1, \int u = 0.$

$k = 3$: $\quad L^2 \otimes \mathbb{R}^3 \xrightarrow{\text{div}, H(\text{div})} L^2 \to 0$

Mixed: $\quad$ Find $\sigma \in H(\text{div}),\ u \in L^2$ s.t.

$$\begin{aligned} \langle \sigma, \tau \rangle - \langle u, \text{div}\, \tau \rangle &= 0, \qquad && \tau \in H(\text{div}), \\ \langle \text{div}\, \sigma, v \rangle &= \langle f, v \rangle, && v \in L^2. \end{aligned}$$

The de Rham complex in 3D

$k = 1$: $\quad L^2 \xrightarrow{\text{grad}, H^1} L^2 \otimes \mathbb{R}^3 \xrightarrow{\text{curl}, H(\text{curl})} L^2 \otimes \mathbb{R}^3$

Mixed: Find $\sigma \in H^1,\ u \in H(\text{curl})$ s.t.

$$\begin{aligned} \langle \sigma, \tau \rangle - \langle u, \operatorname{grad} \tau \rangle &= 0, && \tau \in H^1, \\ \langle \operatorname{grad} \sigma, v \rangle + \langle \operatorname{curl} u, \operatorname{curl} v \rangle &= \langle f, v \rangle, && v \in H(\text{curl}). \end{aligned}$$

$k = 2$: $\quad L^2 \otimes \mathbb{R}^3 \xrightarrow{\text{curl}, H(\text{curl})} L^2 \otimes \mathbb{R}^3 \xrightarrow{\text{div}, H(\text{div})} L^2$

Mixed: Find $\sigma \in H(\text{curl}),\ u \in H(\text{div})$ s.t.

$$\begin{aligned} \langle \sigma, \tau \rangle - \langle u, \operatorname{curl} \tau \rangle &= 0, && \tau \in H(\text{curl}), \\ \langle \operatorname{curl} \sigma, v \rangle + \langle \operatorname{div} u, \operatorname{div} v \rangle &= \langle f, v \rangle, && v \in H(\text{div}). \end{aligned}$$

The Hodge eigenvalue problem

Given the segment

$$W^{k-1} \xrightarrow{d, V^{k-1}} W^k \xrightarrow{d, V^k} W^{k+1}$$

in place of the Hodge Laplacian source problem $\Delta^k u = f$ we can consider the eigenvalue problem:

$$(dd^* + d^* d)u = \lambda u$$

Find nonzero $(\sigma, u) \in V^{k-1} \times V^k$, $\lambda \in \mathbb{R}$ s.t.

$$\begin{aligned} \langle \sigma, \tau \rangle - \langle u, d\tau \rangle &= 0, & \tau &\in V^{k-1}, \\ \langle d\sigma, v \rangle + \langle du, dv \rangle &= \lambda \langle u, v \rangle, & v &\in V^k. \end{aligned}$$

The Hodge heat equation

Or we may consider the Hodge heat equation for $u : [0, T] \to W^k$:

$$\dot{u} + (dd^* + d^*d)u = f, \quad u(0) = u_0$$

Find $\quad (\sigma, u) : [0, T] \to V^{k-1} \times V^k \quad$ s.t.

$$\begin{aligned} \langle \sigma, \tau \rangle \; - \; \langle u, d\tau \rangle &= 0, && \tau \in V^{k-1}, \\ \langle \dot{u}, v \rangle \; + \; \langle d\sigma, v \rangle + \langle du, dv \rangle &= \langle f, v \rangle, && v \in V^k, \end{aligned}$$

The Hodge wave equation

$$\ddot{U} + (dd^* + d^*d)U = 0, \quad U(0) = U_0, \quad \dot{U}(0) = U_1$$

Then $\sigma := d^*U,\ \rho := dU,\ u := \dot{U}$ satisfy

$$\begin{pmatrix} \dot{\sigma} \\ \dot{u} \\ \dot{\rho} \end{pmatrix} + \begin{pmatrix} 0 & -d^* & 0 \\ d & 0 & d^* \\ 0 & -d & 0 \end{pmatrix} \begin{pmatrix} \sigma \\ u \\ \rho \end{pmatrix} = 0$$

Find $\quad (\sigma, u, \rho) : [0, T] \to V^{k-1} \times V^k \times W^{k+1} \quad$ s.t.

$$\begin{aligned} \langle \dot{\sigma}, \tau \rangle \ - \ \langle u, d\tau \rangle &= 0, & \tau &\in V^{k-1}, \\ \langle \dot{u}, v \rangle \ + \ \langle d\sigma, v \rangle + \langle \rho, dv \rangle &= 0, & v &\in V^k, \\ \langle \dot{\rho}, \eta \rangle \ - \ \langle du, \eta \rangle &= 0, & \eta &\in W^{k+1}. \end{aligned}$$

Both the Hodge heat equation and the Hodge wave equation can be shown to be well-posed using the Hille–Yosida–Phillips theory and the results for the Hodge Laplacian.

V. Quenneville-Belair thesis 2015

Example: Maxwell's equations as a Hodge wave equation

$$\dot{D} = \operatorname{curl} H \qquad \dot{B} = -\operatorname{curl} E$$
$$\operatorname{div} D = 0 \qquad \operatorname{div} B = 0$$
$$D = \epsilon E \qquad B = \mu H$$

$$W^0 = L^2(\Omega)$$
$$W^1 = L^2(\Omega, \mathbb{R}^3, \epsilon\, dx)$$
$$W^2 = L^2(\Omega, \mathbb{R}^3, \mu^{-1} dx)$$

$$W^0 \xrightarrow{\operatorname{grad}} W^1 \xrightarrow{-\operatorname{curl}} W^2$$

$(\sigma, E, B) : [0, T] \times \Omega \to \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3$ solves

$$\langle \dot{\sigma}, \tau \rangle - \langle \epsilon E, \operatorname{grad} \tau \rangle = 0 \ \forall \tau,$$
$$\langle \epsilon \dot{E}, F \rangle + \langle \epsilon \operatorname{grad} \sigma, F \rangle - \langle \mu^{-1} B, \operatorname{curl} F \rangle = 0 \ \forall F,$$
$$\langle \mu^{-1} \dot{B}, C \rangle + \langle \mu^{-1} \operatorname{curl} E, C \rangle = 0 \ \forall C.$$

THEOREM

If σ, $\operatorname{div} \epsilon E$, and $\operatorname{div} B$ vanish for $t = 0$, then they vanish for all t, and E, B, $D = \epsilon E$, and $H = \mu^{-1} B$ satisfy Maxwell's equations.

Another complex: the elasticity complex

$$0 \to \underset{\text{displacement}}{L^2 \otimes \mathbb{R}^3} \xrightarrow{\operatorname{sym} \operatorname{grad}} \underset{\text{strain}}{L^2 \otimes \mathbb{S}^3} \xrightarrow{\operatorname{curl} \mathrm{T} \operatorname{curl}} \underset{\text{stress}}{L^2 \otimes \mathbb{S}^3} \xrightarrow{\operatorname{div}} \underset{\text{load}}{L^2 \otimes \mathbb{R}^3} \to 0$$

- $0 \to L^2 \otimes \mathbb{R}^3 \xrightarrow{\operatorname{sym} \operatorname{grad}} L^2 \otimes \mathbb{S}^3$ primal method for elasticity
- $L^2 \otimes \mathbb{S}^3 \xrightarrow{\operatorname{div}} L^2 \otimes \mathbb{R}^3 \to 0$ mixed method for elasticity
- $L^2 \otimes \mathbb{R}^3 \xrightarrow{\operatorname{sym} \operatorname{grad}} L^2 \otimes \mathbb{S}^3 \xrightarrow{\operatorname{curl} \mathrm{T} \operatorname{curl}} L^2 \otimes \mathbb{S}^3$ elastic dislocations

Still other complexes

The Hessian complex:

$\mathbb{R}^{3\times3}$ trace-free

$$0 \to L^2 \xrightarrow{\text{grad grad}} L^2 \otimes \mathbb{S}^3 \xrightarrow{\text{curl}} L^2 \otimes \mathbb{T} \xrightarrow{\text{div}} L^2 \otimes \mathbb{R}^3 \to 0$$

- $0 \to L^2 \xrightarrow{\text{grad grad}} L^2 \otimes \mathbb{S}^3$ primal method for plate equation
- $L^2 \xrightarrow{\text{grad grad}} L^2 \otimes \mathbb{S}^3 \xrightarrow{\text{curl}} L^2 \otimes \mathbb{T}$ Einstein–Bianchi eqs (GR)

Still other complexes

The Hessian complex:

$\mathbb{R}^{3\times 3}$ trace-free

$$0 \to L^2 \xrightarrow{\text{grad grad}} L^2 \otimes \mathbb{S}^3 \xrightarrow{\text{curl}} L^2 \otimes \mathbb{T} \xrightarrow{\text{div}} L^2 \otimes \mathbb{R}^3 \to 0$$

- $0 \to L^2 \xrightarrow{\text{grad grad}} L^2 \otimes \mathbb{S}^3$ primal method for plate equation
- $L^2 \xrightarrow{\text{grad grad}} L^2 \otimes \mathbb{S}^3 \xrightarrow{\text{curl}} L^2 \otimes \mathbb{T}$ Einstein–Bianchi eqs (GR)

2D Stokes complex:

$$0 \to H^2 \xrightarrow{\text{curl}} H^1 \otimes \mathbb{R}^2 \xrightarrow{\text{div}} L^2 \to 0$$

Still other complexes

The Hessian complex:

$\mathbb{R}^{3\times 3}$ trace-free

$$0 \to L^2 \xrightarrow{\text{grad grad}} L^2 \otimes \mathbb{S}^3 \xrightarrow{\text{curl}} L^2 \otimes \mathbb{T} \xrightarrow{\text{div}} L^2 \otimes \mathbb{R}^3 \to 0$$

- $0 \to L^2 \xrightarrow{\text{grad grad}} L^2 \otimes \mathbb{S}^3$ primal method for plate equation
- $L^2 \xrightarrow{\text{grad grad}} L^2 \otimes \mathbb{S}^3 \xrightarrow{\text{curl}} L^2 \otimes \mathbb{T}$ Einstein–Bianchi eqs (GR)

2D Stokes complex:

$$0 \to H^2 \xrightarrow{\text{curl}} H^1 \otimes \mathbb{R}^2 \xrightarrow{\text{div}} L^2 \to 0$$

3D Stokes complex:

$$0 \to H^2 \xrightarrow{\text{grad}} H^1(\text{curl}) \xrightarrow{\text{curl}} H^1 \otimes \mathbb{R}^3 \xrightarrow{\text{div}} L^2 \to 0$$

Evans 2011, Falk–Neilan 2013

Structure-preserving discretization of Hilbert complexes

Structure-preserving discretization

For discretization we choose subspaces $V_h^{k-1} \subset V^{k-1}, V_h^k \subset V^k$ and use *Galerkin's method:*

Find $(\sigma_h, u_h, p_h) \in V_h^{k-1} \times V_h^k \times \mathfrak{H}^h$ s.t.

$$\begin{aligned} \langle \sigma_h, \tau \rangle - \langle u_h, d\tau \rangle &= 0, && \tau \in V_h^{k-1}, \\ \langle d\sigma_h, v \rangle + \langle du_h, dv \rangle + \langle p_h, v \rangle &= \langle f, v \rangle, && v \in V_h^k, \\ \langle u_h, q \rangle &= 0, && q \in \mathfrak{H}_h. \end{aligned}$$

where $d_h = d|_{V_h}, \mathfrak{Z}_h = \mathcal{N}(d_h), \mathfrak{B}_h = \mathcal{R}(d_h), \mathfrak{H}_h = \mathfrak{Z}_h \cap \mathfrak{B}_h^\perp$

When is this approximation stable, consistent, and convergent?

Assumptions on the discretization

Besides good approximation properties, the key requirements are structural:

Subcomplex assumption: $d\,V_h^k \subset V_h^{k+1}$

Bounded Cochain Projection assumption: $\exists\, \pi_h^k : V^k \to V_h^k$

$$\begin{tikzcd} V^k \arrow[r, "d^k"] \arrow[d, "\pi_h^k"] & V^{k+1} \arrow[d, "\pi_h^{k+1}"] \\ V_h^k \arrow[r, "d^k"] & V_h^{k+1} \end{tikzcd}$$

- π_h^k is bounded, uniformly in h
- $\pi_h^{k+1} d^k = d^k \pi_h^k$
- π_h^k preserves V_h^k

The subcomplex property implies that $V_h^{k-1} \xrightarrow{d_h} V_h^k \xrightarrow{d_h} W^2$ is itself an H-complex. So it has its own harmonic forms, Hodge decomposition, and Poincaré inequality. The Galerkin method is precisely the Hodge Laplacian for the discrete complex.

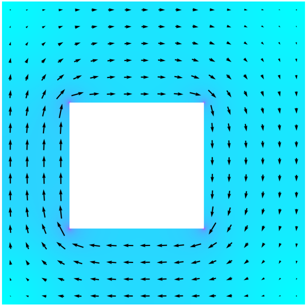
Consequences of the assumptions

Theorem

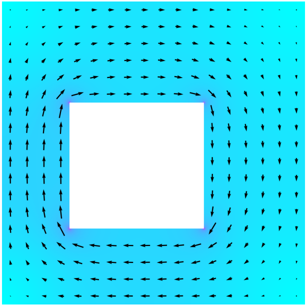
Given the approximation, subcomplex, and BCP assumptions:

- $\mathfrak{H} \cong \mathfrak{H}_h$ *and* $\operatorname{gap}(\mathfrak{H}, \mathfrak{H}_h) \to 0$.
- *The Galerkin method is consistent.*
- *The discrete Poincaré inequality* $\|\omega\| \le c\|d\omega\|, \quad \omega \in \mathfrak{Z}_h^{k\perp}$, *holds with c independent of h.*
- *The Galerkin method is stable.*
- *The Galerkin method is convergent with quasioptimal error estimates.*

Example: eigenvalues of the 1-form Laplacian



Example: eigenvalues of the 1-form Laplacian



primal formulation with Lagrange finite elements
$$(\operatorname{div} u, \operatorname{div} v) + (\operatorname{curl} u, \operatorname{curl} v) = \lambda(u, v)$$

	Degree 1		Degree 3	
# Elements	λ_1	λ_2	λ_1	λ_2
256	2.270	2.360	1.896	1.970
1,024	2.050	2.132	1.854	1.925
4,096	1.940	2.016	1.828	1.897
16,384	1.879	1.952	1.812	1.880
65,536	1.843	1.914	1.802	1.870
262,144	1.821	1.890	1.796	1.863

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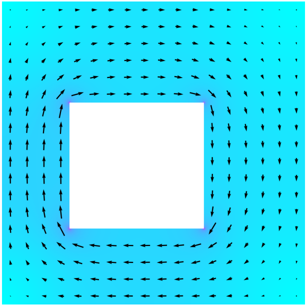

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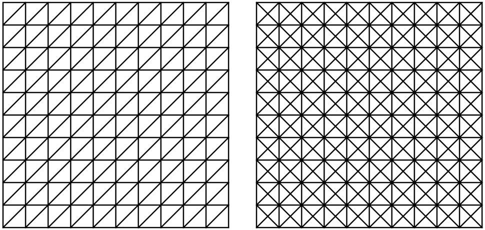
65,536	1.843	1.914	1.802	1.870
262,144	1.821	1.890	1.796	1.863

mixed formulation and structure-preserving elements

	Degree 1		Degree 3	
# Elements	λ_1	λ_2	λ_1	λ_2
256	0.000	0.638	0.000	0.619
1,024	0.000	0.625	0.000	0.618
4,096	0.000	0.620	0.000	0.617
16,384	0.000	0.618	0.000	0.617
65,536	0.000	0.618	0.000	0.617
262,144	0.000	0.617	0.000	0.617

Example: Maxwell eigenvalue problem



First 12 Maxwell eigenvalues and Galerkin approximations of them.

Exact	1	1	2	4	4	5	5	8	9	9	10	10
Diagonal mesh												
Lagrange	5.16	5.26	5.26	5.30	5.39	5.45	5.53	5.61	5.61	5.62	5.71	5.73
FEEC	1.00	1.00	2.00	4.00	4.00	5.00	5.00	8.01	8.98	8.99	9.99	9.99
Crisscross mesh												
Lagrange	1.00	1.00	2.00	4.00	4.00	5.00	5.00	6.00	8.01	9.01	9.01	10.02
FEEC	1.00	1.00	2.00	4.00	4.00	5.00	5.00	7.99	9.00	9.00	10.00	10.00

Structure-preserving finite elements

The construction of finite element spaces satisfying the subcomplex and BCP properties varies according to the complex.

For the de Rham complex it depends on the structure of differential forms:

- wedge product
- exterior derivative
- form integration
- pullbacks
- Stokes's theorem
- the Koszul differential κ
- the homotopy property: $(d\kappa + \kappa d)u = (r+k)u, \quad u \in \mathcal{H}_r\Lambda^k$

Finite element differential forms on simplicial meshes

A primary conclusion of FEEC is that in every dimension n there are *two* natural spaces of finite element differential forms associated to each simplicial mesh $\mathcal{T}_h$, each form degree k, and each polynomial polynomial degree r:

- The spaces $\mathcal{P}_r\Lambda^k(\mathcal{T}_h)$ which form a de Rham subcomplex with decreasing degree:

$$0 \to \mathcal{P}_r\Lambda^0(\mathcal{T}_h) \xrightarrow{d} \mathcal{P}_{r-1}\Lambda^1(\mathcal{T}_h) \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{P}_{r-n}\Lambda^n(\mathcal{T}_h) \to 0$$

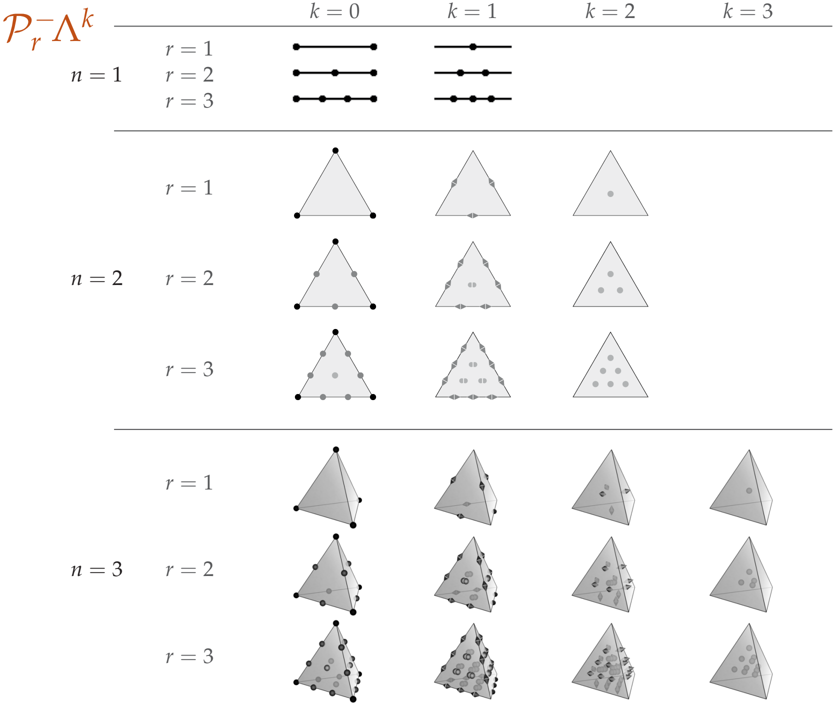
- The spaces $\mathcal{P}_r^-\Lambda^k(\mathcal{T}_h)$ which form a de Rham subcomplex with constant degree:

$$0 \to \mathcal{P}_r^-\Lambda^0(\mathcal{T}_h) \xrightarrow{d} \mathcal{P}_r^-\Lambda^1(\mathcal{T}_h) \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{P}_r^-\Lambda^n(\mathcal{T}_h) \to 0$$

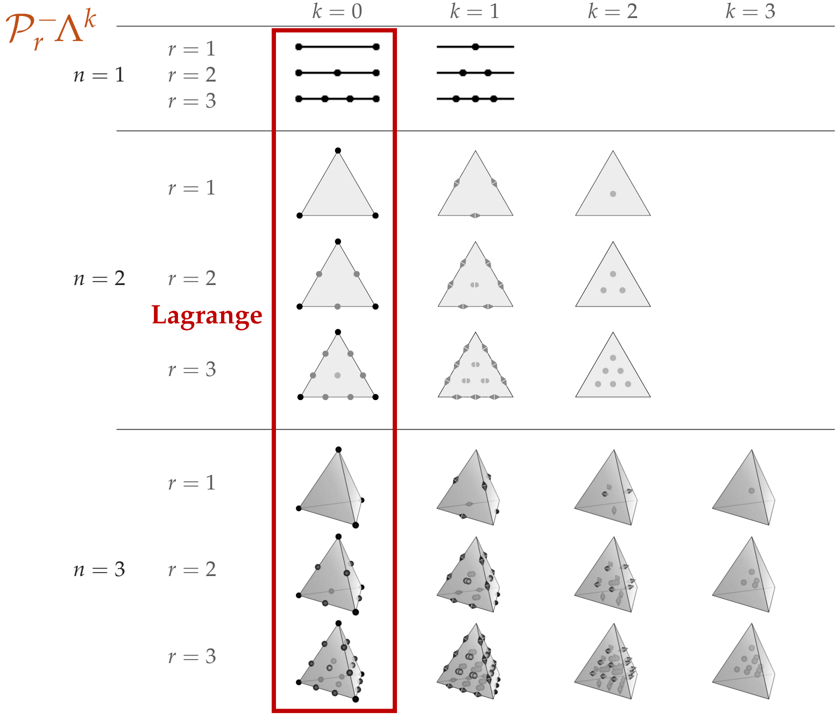
Pairs of spaces which satisfy the subcomplex property and BCP property can be selected from these in *four* ways:

$$\mathcal{P}_r\Lambda^{k-1}(\mathcal{T}_h) \times \mathcal{P}_{r-1}\Lambda^k(\mathcal{T}_h) \qquad \mathcal{P}_r\Lambda^{k-1}(\mathcal{T}_h) \times \mathcal{P}_r^-\Lambda^k(\mathcal{T}_h)$$

$$\mathcal{P}_r^-\Lambda^{k-1}(\mathcal{T}_h) \times \mathcal{P}_{r-1}\Lambda^k(\mathcal{T}_h) \qquad \mathcal{P}_r^-\Lambda^{k-1}(\mathcal{T}_h) \times \mathcal{P}_r^-\Lambda^k(\mathcal{T}_h)$$



$\mathcal{P}_r^-\Lambda^k$
$k = 0$
$k = 1$
$k = 2$
$k = 3$
$n = 1$
$n = 2$
$n = 3$
$r = 1$
$r = 2$
$r = 3$



$\mathcal{P}_r^-\Lambda^k$
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$k = 1$
$k = 2$
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$n = 1$
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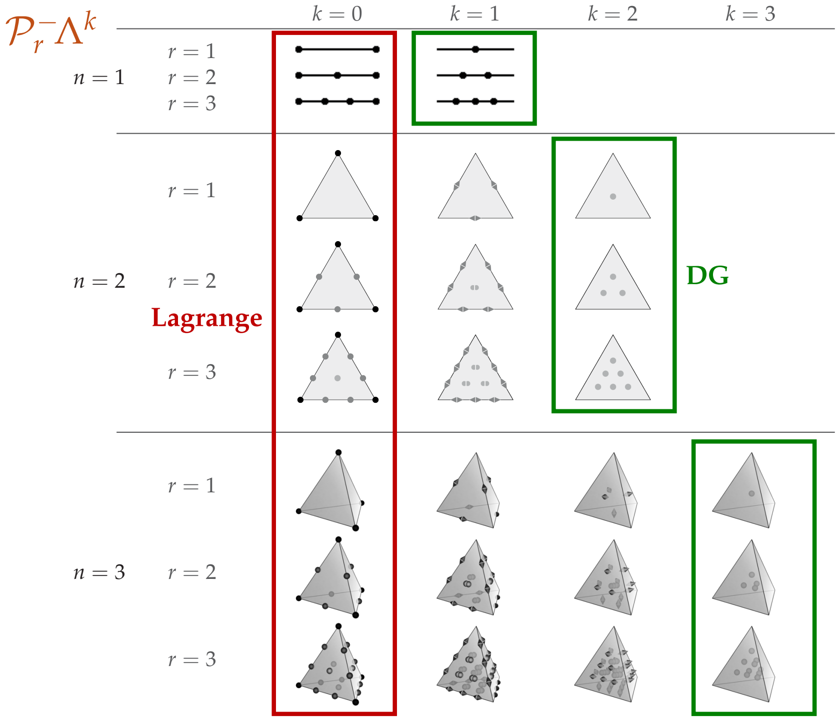
$n = 3$
$r = 1$
$r = 2$
$r = 3$
Lagrange



$\mathcal{P}_r^-\Lambda^k$
$k=0$
$k=1$
$k=2$
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$r=1$
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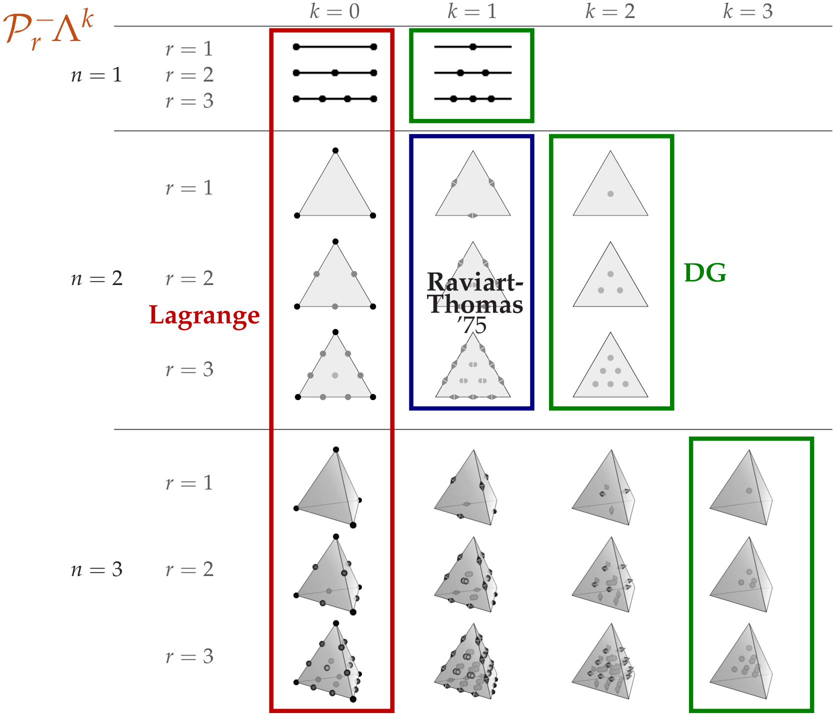
$r=3$
Lagrange
DG


$\mathcal{P}_r^-\Lambda^k$
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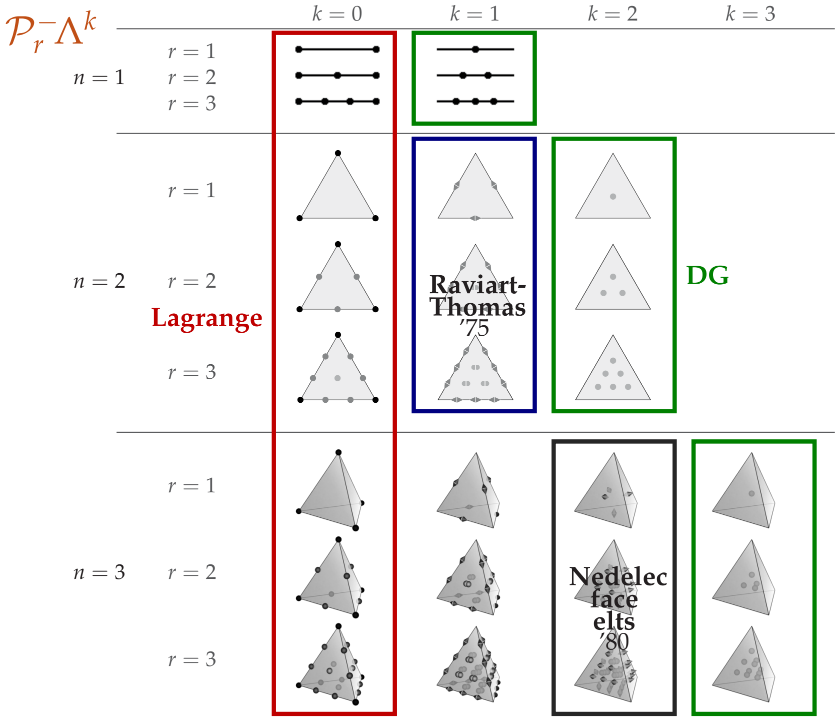
Lagrange
Raviart-Thomas '75
DG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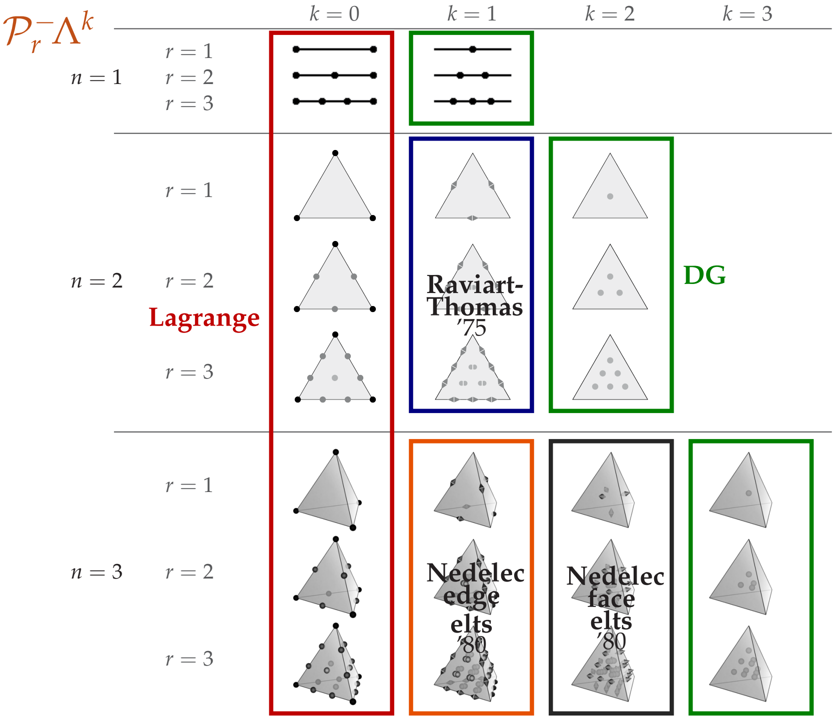
$r=3$
Lagrange
Raviart-Thomas '75
DG
Nedelec face elts '80



$\mathcal{P}_r^-\Lambda^k$
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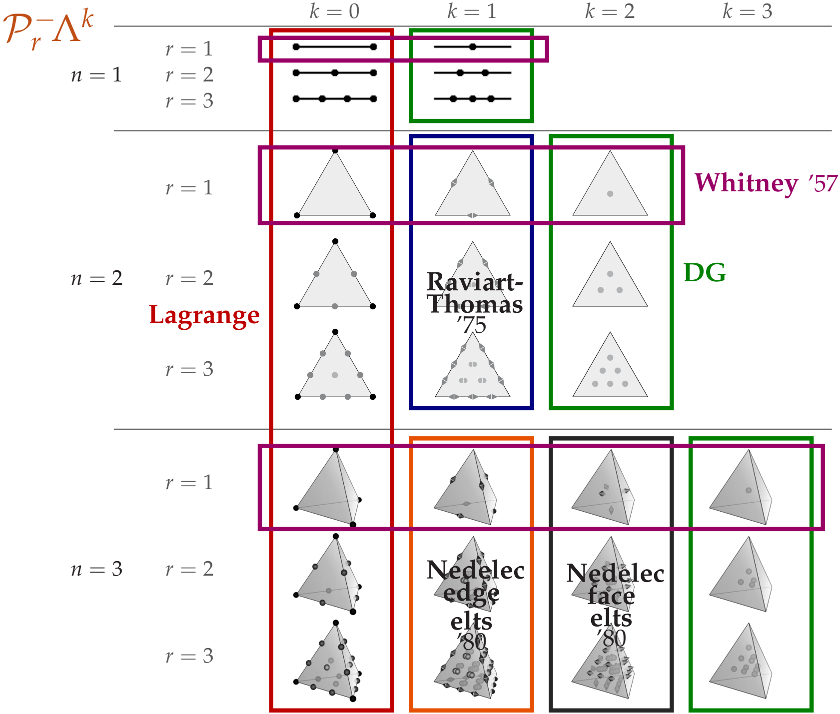
Lagrange
Raviart-Thomas '75
DG
Nedelec edge elts '80
Nedelec face elts '80


$\mathcal{P}_r^-\Lambda^k$
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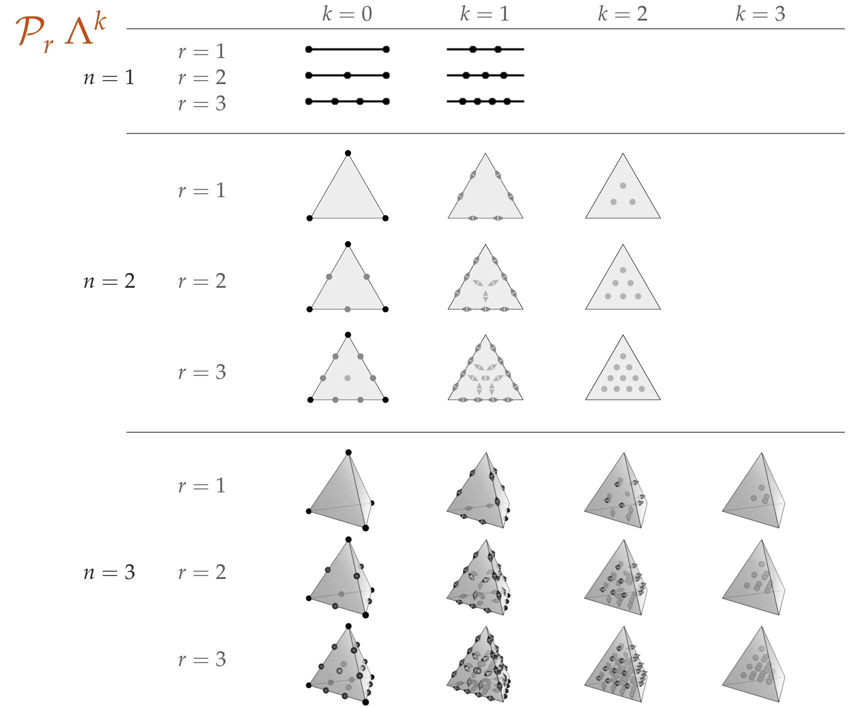
Whitney '57
DG
Lagrange
Raviart-Thomas '75
Nedelec edge elts '80
Nedelec face elts '80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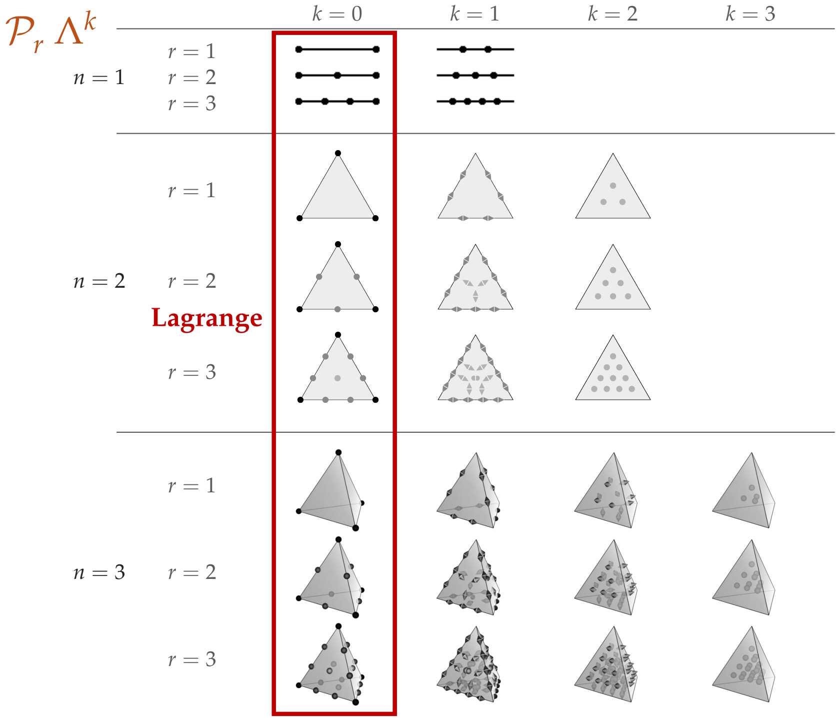

$\mathcal{P}_r \Lambda^k$
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Lagrange
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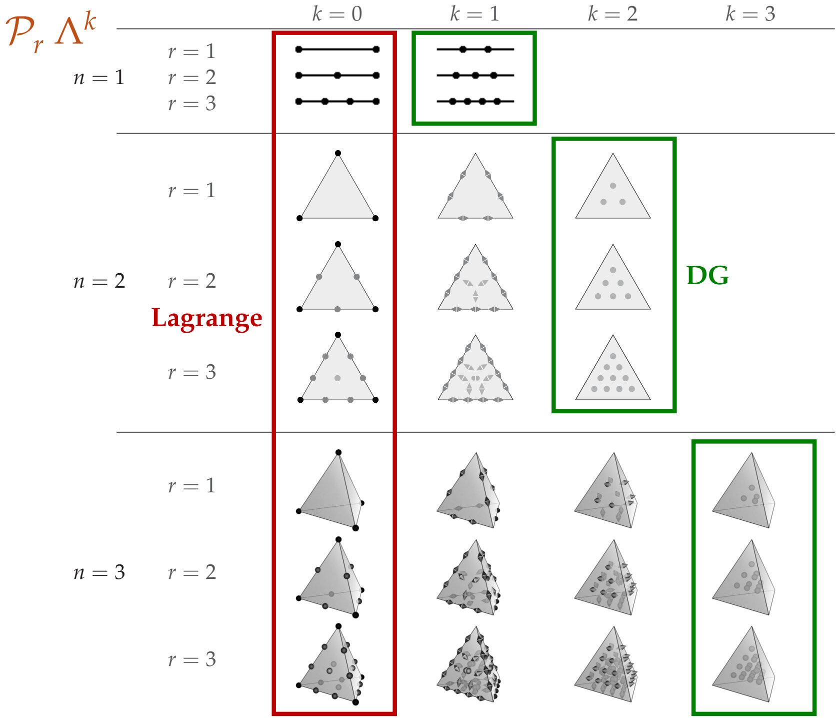

$\mathcal{P}_r\Lambda^k$
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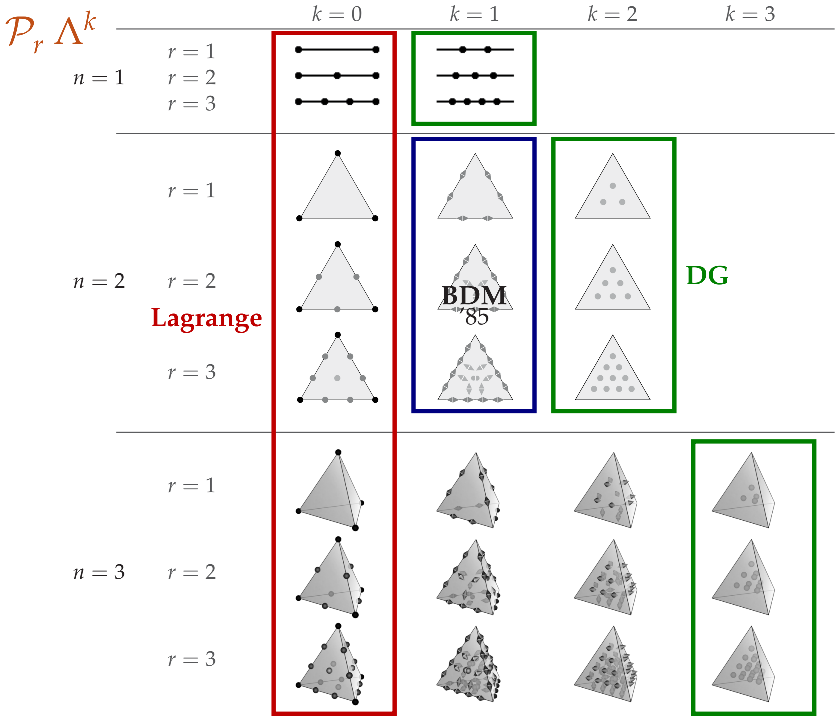
$r=3$
Lagrange
DG



$\mathcal{P}_r \Lambda^k$
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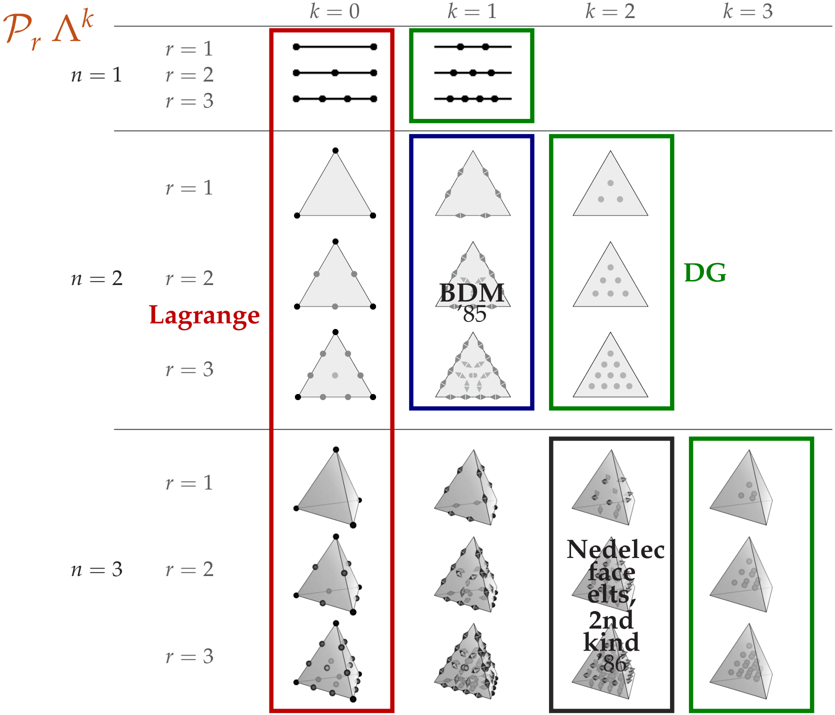
Lagrange
BDM
'85
DG
$n = 3$
$r = 1$
$r = 2$
$r = 3$



$\mathcal{P}_r \Lambda^k$
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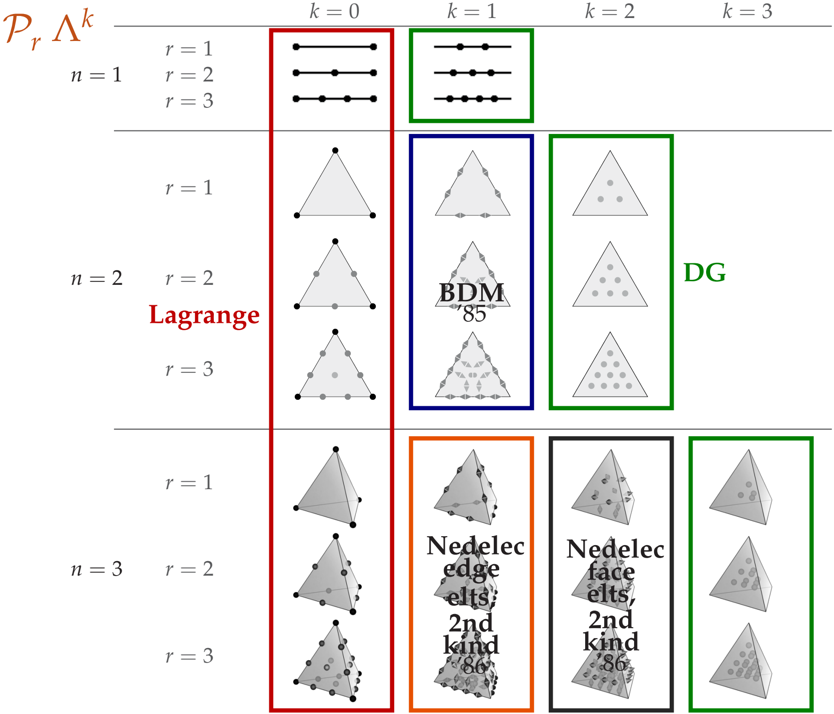
$r = 3$
Lagrange
BDM '85
DG
Nedelec face elts, 2nd kind '86



$\mathcal{P}_r \Lambda^k$
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$k = 3$
$n = 1$
$n = 2$
$n = 3$
$r = 1$
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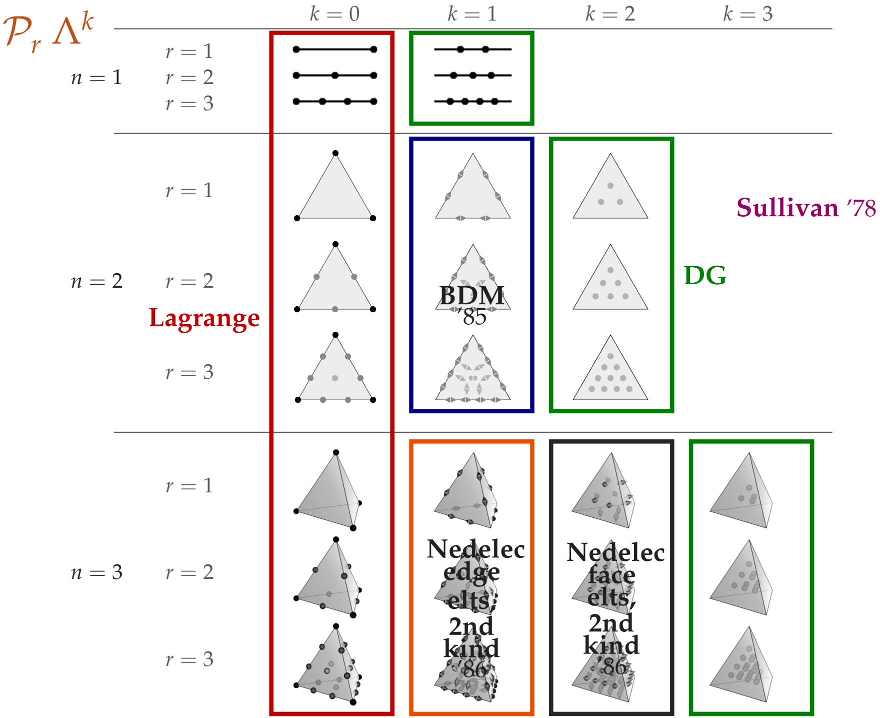
Lagrange
BDM '85
DG
Nedelec edge elts, 2nd kind '86
Nedelec face elts, 2nd kind '86



$\mathcal{P}_r \Lambda^k$
$k=0$
$k=1$
$k=2$
$k=3$
$n=1$
$n=2$
$n=3$
$r=1$
$r=2$
$r=3$
Lagrange
BDM '85
DG
Sullivan '78
Nedelec edge elts, 2nd kind '86
Nedelec face elts, 2nd kind '86

Periodic Table of the Finite Elements

$\mathcal{P}_r^-\Lambda^k$

The shape function space for $\mathcal{P}_r^-\Lambda^k$ is

$$\mathcal{P}_{r-1}\Lambda^k + \kappa\mathcal{P}_{r-1}\Lambda^{k+1},$$

where κ is the Koszul differential.[7] It includes the full polynomial space $\mathcal{P}_{r-1}\Lambda^k$, is included in $\mathcal{P}_r\Lambda^k$, and has dimension

$$\dim \mathcal{P}_r^-\Lambda^k(\Delta_n) = \binom{r+n}{r+k}\binom{r+k-1}{k}.$$

The degrees of freedom are given on faces f of dimension $d \geq k$ by moments of the trace weighted by a full polynomial space:

$$u \mapsto \int_f (\mathrm{tr}_f\, u) \wedge q, \quad q \in \mathcal{P}_{r+k-d-1}\Lambda^{d-k}(f).$$

The spaces with constant degree r form a complex:

$$\mathcal{P}_r^-\Lambda^0 \xrightarrow{d} \mathcal{P}_r^-\Lambda^1 \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{P}_r^-\Lambda^n.$$

$\mathcal{P}_r\Lambda^k$

The shape function space for $\mathcal{P}_r\Lambda^k$ consists of all differential k-forms with polynomial coefficients of degree at most r, and has dimension

$$\dim \mathcal{P}_r\Lambda^k(\Delta_n) = \binom{r+n}{r+k}\binom{r+k}{k}.$$

The degrees of freedom are given on faces f of dimension $d \geq k$ by moments of the trace weighted by a $\mathcal{P}_r^-$ space:

$$u \mapsto \int_f (\mathrm{tr}_f\, u) \wedge q, \quad q \in \mathcal{P}^-_{r+k-d}\Lambda^{d-k}(f).$$

The spaces with decreasing degree r form a complex:

$$\mathcal{P}_r\Lambda^0 \xrightarrow{d} \mathcal{P}_{r-1}\Lambda^1 \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{P}_{r-n}\Lambda^n.$$

$\mathcal{Q}_r^-\Lambda^k$

This family is constructed from the complex of 1-dimensional finite elements using a tensor product construction.[10] The shape function space on the unit cube $\square_n = I^n$ is given by

$$\bigoplus_{\sigma\in\Sigma(k,n)} \left[\bigotimes_{i=1}^{n} \mathcal{P}_{r-\delta_{i,\sigma}}(I)\right] dx^{\sigma_1}\wedge\cdots\wedge dx^{\sigma_k},$$

where $\Sigma(k,n)$ denotes the increasing maps $\{1,\ldots,k\} \to \{1,\ldots,n\}$. Its dimension is $\dim \mathcal{Q}_r^-\Lambda^k(\square_n) = \binom{n}{k} r^k (r+1)^{n-k}$. The degrees of freedom are

$$u \mapsto \int_f (\mathrm{tr}_f\, u) \wedge q, \quad q \in \mathcal{Q}^-_{r-1}\Lambda^{d-k}(f).$$

The spaces with constant degree r form a complex:

$$\mathcal{Q}_r^-\Lambda^0 \xrightarrow{d} \mathcal{Q}_r^-\Lambda^1 \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{Q}_r^-\Lambda^n.$$

$\mathcal{S}_r\Lambda^k$

The shape function space for $\mathcal{S}_r\Lambda^k$ is given by

$$\mathcal{P}_r\Lambda^k \oplus \bigoplus_{\ell\geq 1}\left[\kappa\mathcal{H}_{r+\ell-1,\ell}\Lambda^{k+1} \oplus d\kappa\mathcal{H}_{r+\ell,\ell}\Lambda^k\right],$$

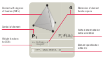
where $\mathcal{H}_{r,\ell}\Lambda^k$ consists of homogeneous polynomial k-forms of degree r which are linear and undifferentiated in at least ℓ variables.[11] Its dimension is $\dim \mathcal{S}_r\Lambda^k(\square_n) = \sum_{d\geq k} 2^{n-d}\binom{n}{d}\binom{r-d+2k}{d}\binom{d}{k}$. The degrees of freedom are

$$u \mapsto \int_f (\mathrm{tr}_f\, u) \wedge q, \quad q \in \mathcal{P}_{r-2(d-k)}\Lambda^{d-k}(f).$$

The spaces with decreasing degree r form a complex:

$$\mathcal{S}_r\Lambda^0 \xrightarrow{d} \mathcal{S}_{r-1}\Lambda^1 \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{S}_{r-n}\Lambda^n.$$

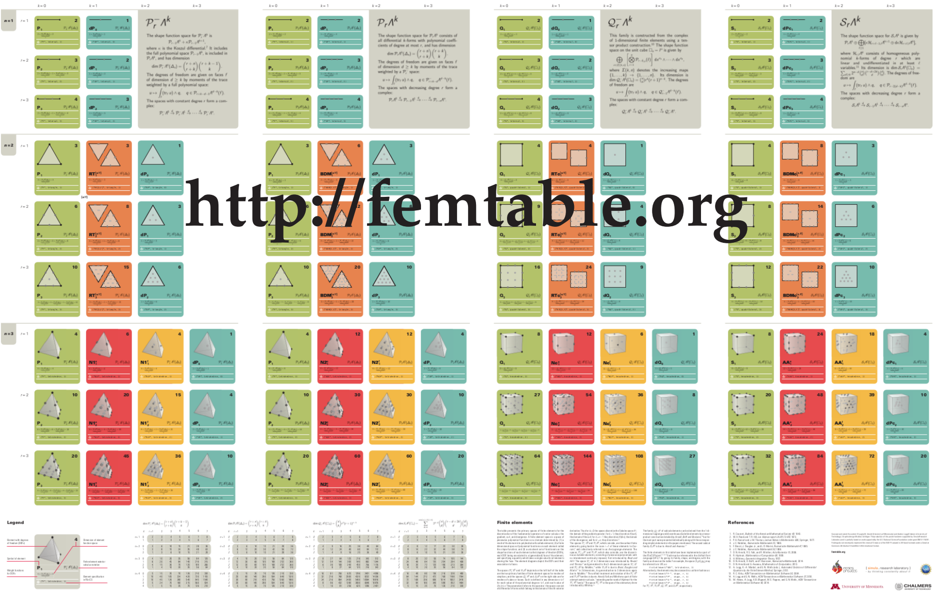
Legend



Finite elements

References

Periodic Table of the Finite Elements



New complexes from old

Elasticity with weak symmetry

The mixed formulation of elasticity with *weak symmetry* is more amenable to discretization than the standard mixed formulation.

Fraeijs de Veubeke '75

$$p = \operatorname{skw}\operatorname{grad} u, \quad A\sigma = \operatorname{grad} u - p$$

Find $\quad \sigma \in L^2(\Omega) \otimes \mathbb{R}^{n\times n}, u \in L^2(\Omega) \otimes \mathbb{R}^n, p \in L^2(\Omega) \otimes \mathbb{R}^{n\times n}_{\mathrm{skw}} \quad$ s.t.

$$\begin{aligned} \langle A\sigma, \tau\rangle + \langle u, \operatorname{div}\tau\rangle + \langle p, \tau\rangle &= 0, & \tau &\in L^2(\Omega) \otimes \mathbb{R}^{n\times n} \\ -\langle \operatorname{div}\sigma, v\rangle &= \langle f, v\rangle, & v &\in L^2(\Omega) \otimes \mathbb{R}^n \\ -\langle \sigma, q\rangle &= 0, & q &\in L^2(\Omega) \otimes \mathbb{R}^{n\times n}_{\mathrm{skw}} \end{aligned}$$

This is exactly the mixed Hodge Laplacian for the complex:

$$L^2_A(\Omega) \otimes \mathbb{R}^{n\times n} \xrightarrow{(-\operatorname{div}, -\operatorname{skw})} [L^2(\Omega) \otimes \mathbb{R}^n] \oplus [L^2(\Omega) \otimes \mathbb{R}^{n\times n}_{\mathrm{skw}}] \longrightarrow 0$$

Well-posedness

$$L^2_A(\Omega) \otimes \mathbb{R}^{n\times n} \xrightarrow{(-\operatorname{div}, -\operatorname{skw})} [L^2(\Omega) \otimes \mathbb{R}^n] \oplus [L^2(\Omega) \otimes \mathbb{R}^{n\times n}_{\mathrm{skw}}] \longrightarrow 0$$

Well-posedness depends on the exactness of the complex. This can be shown by relating the complex to two de Rham complexes:

$$\begin{tikzcd}
 & L^2(\Omega) \otimes \mathbb{R}^n \otimes \mathbb{R}^{n\times n}_{\mathrm{skw}} \arrow[r, "\operatorname{div}"] & L^2(\Omega) \otimes \mathbb{R}^{n\times n}_{\mathrm{skw}} \arrow[r] & 0 \\
L^2(\Omega) \otimes \mathbb{R}^{n\times n} \arrow[ur, "S"] \arrow[r, "\operatorname{curl}"] & L^2(\Omega) \otimes \mathbb{R}^{n\times n} \arrow[ur, "-\operatorname{skw}"] \arrow[r, "-\operatorname{div}"] & L^2(\Omega) \otimes \mathbb{R}^n \arrow[r] & 0
\end{tikzcd}$$

$S\tau = \tau^T - \operatorname{tr}(\tau) I$ (invertible)

$$L^2_A(\Omega) \otimes \mathbb{R}^{n\times n} \xrightarrow{(-\operatorname{div}, -\operatorname{skw})} [L^2(\Omega) \otimes \mathbb{R}^n] \oplus [L^2(\Omega) \otimes \mathbb{R}^{n\times n}_{\mathrm{skw}}] \longrightarrow 0$$

Well-posedness depends on the exactness of the complex. This can be shown by relating the complex to two de Rham complexes:

$$\begin{tikzcd}
 & & q & \\
 & L^2(\Omega) \otimes \mathbb{R}^n \otimes \mathbb{R}^{n\times n}_{\mathrm{skw}} \arrow[r, "\operatorname{div}"] & L^2(\Omega) \otimes \mathbb{R}^{n\times n}_{\mathrm{skw}} \arrow[r] & 0 \\
L^2(\Omega) \otimes \mathbb{R}^{n\times n} \arrow[ur, "S"] \arrow[r, "\operatorname{curl}"] & L^2(\Omega) \otimes \mathbb{R}^{n\times n} \arrow[ur, "-\operatorname{skw}"] \arrow[r, "-\operatorname{div}"] & L^2(\Omega) \otimes \mathbb{R}^n \arrow[r] & 0 \\
 & & v &
\end{tikzcd}$$

$S\tau = \tau^T - \operatorname{tr}(\tau) I$ (invertible)

Well-posedness

$$L^2_A(\Omega) \otimes \mathbb{R}^{n\times n} \xrightarrow{(-\operatorname{div}, -\operatorname{skw})} [L^2(\Omega) \otimes \mathbb{R}^n] \oplus [L^2(\Omega) \otimes \mathbb{R}^{n\times n}_{\operatorname{skw}}] \longrightarrow 0$$

Well-posedness depends on the exactness of the complex. This can be shown by relating the complex to two de Rham complexes:

$$\begin{tikzcd}
 & & q & \\
 & L^2(\Omega) \otimes \mathbb{R}^n \otimes \mathbb{R}^{n\times n}_{\operatorname{skw}} \arrow[r, "\operatorname{div}"] & L^2(\Omega) \otimes \mathbb{R}^{n\times n}_{\operatorname{skw}} \arrow[r] & 0 \\
L^2(\Omega) \otimes \mathbb{R}^{n\times n} \arrow[ur, "S"] \arrow[r, "\operatorname{curl}"] & L^2(\Omega) \otimes \mathbb{R}^{n\times n} \arrow[ur, "-\operatorname{skw}"] \arrow[r, "-\operatorname{div}"] & L^2(\Omega) \otimes \mathbb{R}^n \arrow[r] & 0 \\
 & \rho & v \arrow[l] &
\end{tikzcd}$$

$S\tau = \tau^T - \operatorname{tr}(\tau)I$ (invertible)

Well-posedness

$$L^2_A(\Omega) \otimes \mathbb{R}^{n\times n} \xrightarrow{(-\operatorname{div}, -\operatorname{skw})} [L^2(\Omega) \otimes \mathbb{R}^n] \oplus [L^2(\Omega) \otimes \mathbb{R}^{n\times n}_{\operatorname{skw}}] \longrightarrow 0$$

Well-posedness depends on the exactness of the complex. This can be shown by relating the complex to two de Rham complexes:

$$\begin{tikzcd}
 & & q + \operatorname{skw} \rho & \\
 & L^2(\Omega) \otimes \mathbb{R}^n \otimes \mathbb{R}^{n\times n}_{\operatorname{skw}} \arrow[r, "\operatorname{div}"] & L^2(\Omega) \otimes \mathbb{R}^{n\times n}_{\operatorname{skw}} \arrow[r] & 0 \\
L^2(\Omega) \otimes \mathbb{R}^{n\times n} \arrow[ur, "S"] \arrow[r, "\operatorname{curl}"] & L^2(\Omega) \otimes \mathbb{R}^{n\times n} \arrow[ur, "-\operatorname{skw}"] \arrow[r, "-\operatorname{div}"] & L^2(\Omega) \otimes \mathbb{R}^n \arrow[r] & 0 \\
 & \rho \arrow[uuur] & v \arrow[l] &
\end{tikzcd}$$

$S\tau = \tau^T - \operatorname{tr}(\tau) I$ (invertible)

$$L^2_A(\Omega) \otimes \mathbb{R}^{n\times n} \xrightarrow{(-\operatorname{div}, -\operatorname{skw})} [L^2(\Omega) \otimes \mathbb{R}^n] \oplus [L^2(\Omega) \otimes \mathbb{R}^{n\times n}_{\operatorname{skw}}] \longrightarrow 0$$

Well-posedness depends on the exactness of the complex. This can be shown by relating the complex to two de Rham complexes:

$$\begin{tikzcd}
 & \psi & q + \operatorname{skw}\rho \arrow[l] & \\
 & L^2(\Omega) \otimes \mathbb{R}^n \otimes \mathbb{R}^{n\times n}_{\operatorname{skw}} \arrow[r, "\operatorname{div}"] & L^2(\Omega) \otimes \mathbb{R}^{n\times n}_{\operatorname{skw}} \arrow[r] & 0 \\
L^2(\Omega) \otimes \mathbb{R}^{n\times n} \arrow[ur, "S"] \arrow[r, "\operatorname{curl}"] & L^2(\Omega) \otimes \mathbb{R}^{n\times n} \arrow[ur, "-\operatorname{skw}"] \arrow[r, "-\operatorname{div}"] & L^2(\Omega) \otimes \mathbb{R}^n \arrow[r] & 0 \\
 & \rho \arrow[uur] & v \arrow[l] &
\end{tikzcd}$$

$S\tau = \tau^T - \operatorname{tr}(\tau)I \quad$ (invertible)

Well-posedness

$$L^2_A(\Omega)\otimes\mathbb{R}^{n\times n} \xrightarrow{(-\operatorname{div},-\operatorname{skw})} [L^2(\Omega)\otimes\mathbb{R}^n]\oplus[L^2(\Omega)\otimes\mathbb{R}^{n\times n}_{\operatorname{skw}}] \longrightarrow 0$$

Well-posedness depends on the exactness of the complex. This can be shown by relating the complex to two de Rham complexes:

$$\begin{tikzcd}
 & & \psi \arrow[dddll] & & q+\operatorname{skw}\rho \arrow[ll] & \\
 & & L^2(\Omega)\otimes\mathbb{R}^n\otimes\mathbb{R}^{n\times n}_{\operatorname{skw}} \arrow[rr, "\operatorname{div}"] & & L^2(\Omega)\otimes\mathbb{R}^{n\times n}_{\operatorname{skw}} \arrow[r] & 0 \\
L^2(\Omega)\otimes\mathbb{R}^{n\times n} \arrow[rr, "\operatorname{curl}"] \arrow[urr, "S"] & & L^2(\Omega)\otimes\mathbb{R}^{n\times n} \arrow[rr, "-\operatorname{div}"] \arrow[urr, "-\operatorname{skw}"] & & L^2(\Omega)\otimes\mathbb{R}^n \arrow[r] & 0 \\
\phi & & \rho \arrow[uuurr] & & v \arrow[ll] &
\end{tikzcd}$$

$S\tau = \tau^T - \operatorname{tr}(\tau)I \quad$ (invertible)

$$L^2_A(\Omega) \otimes \mathbb{R}^{n\times n} \xrightarrow{(-\operatorname{div}, -\operatorname{skw})} [L^2(\Omega) \otimes \mathbb{R}^n] \oplus [L^2(\Omega) \otimes \mathbb{R}^{n\times n}_{\mathrm{skw}}] \longrightarrow 0$$

Well-posedness depends on the exactness of the complex. This can be shown by relating the complex to two de Rham complexes:

$$\begin{tikzcd}
 & \psi \arrow[dddl] & & q + \operatorname{skw}\rho \arrow[ll] & & \\
 & L^2(\Omega) \otimes \mathbb{R}^n \otimes \mathbb{R}^{n\times n}_{\mathrm{skw}} \arrow[rr, "\operatorname{div}"] & & L^2(\Omega) \otimes \mathbb{R}^{n\times n}_{\mathrm{skw}} \arrow[rr] & & 0 \\
L^2(\Omega) \otimes \mathbb{R}^{n\times n} \arrow[ur, "S"] \arrow[rr, "\operatorname{curl}"] & & L^2(\Omega) \otimes \mathbb{R}^{n\times n} \arrow[ur, "-\operatorname{skw}"] \arrow[rr, "-\operatorname{div}"] & & L^2(\Omega) \otimes \mathbb{R}^n \arrow[r] & 0 \\
\phi \arrow[rr] & & \operatorname{curl}\phi + \rho \arrow[uuur] & & v \arrow[ll] &
\end{tikzcd}$$

$S\tau = \tau^T - \operatorname{tr}(\tau) I$ (invertible)

Discretization

To discretize we select discrete de Rham subcomplexes with commuting projections

$$\tilde V_h^0 \xrightarrow{\text{curl}} \tilde V_h^1 \xrightarrow{-\text{div}} \tilde V_h^2 \to 0, \qquad V_h^1 \xrightarrow{-\text{div}} V_h^2 \to 0$$

to get the discrete complex

$$\tilde V_h^1 \otimes \mathbb{R}^n \xrightarrow{(-\operatorname{div}, -\operatorname{skw})} (\tilde V_h^2 \otimes \mathbb{R}^n) \times (V_h^2 \otimes \mathbb{R}^{n\times n}_{\text{skw}}) \to 0$$

We get stability if we can carry out the diagram chase on:

$$\begin{tikzcd}
 & V_h^1 \otimes \mathbb{R}^{n\times n}_{\text{skw}} \arrow[r, "\text{div}"] & V_h^2 \otimes \mathbb{R}^{n\times n}_{\text{skw}} \arrow[r] & 0 \\
\tilde V_h^0 \otimes \mathbb{R}^n \arrow[ur, "\pi_h^1 S"] \arrow[r, "\text{curl}"] & \tilde V_h^1 \otimes \mathbb{R}^n \arrow[ur, "-\pi_h^2\,\text{skw}"] \arrow[r, "-\text{div}"] & \tilde V_h^2 \otimes \mathbb{R}^n \arrow[r] & 0
\end{tikzcd}$$

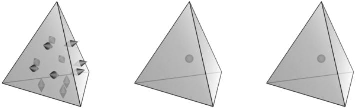
This requires that $\pi_h^1 S : \tilde V_h^0 \otimes \mathbb{R}^n \to V_h^1 \otimes \mathbb{R}^{n\times n}_{\text{skw}}$ is *surjective*.

Stable elements

The requirement that $\pi_h^1 S : \tilde{V}_h^0 \otimes \mathbb{R}^n \to V_h^1 \otimes \mathbb{R}^{n\times n}_{\mathrm{skw}}$ is surjective can be checked by looking at DOFs.
The simplest choice is

$$\mathcal{P}_{r+1}\Lambda^{n-2} \xrightarrow{\mathrm{curl}} \mathcal{P}_r\Lambda^{n-1} \xrightarrow{-\mathrm{div}} \mathcal{P}_{r-1}\Lambda^n \to 0, \quad \mathcal{P}_r^-\Lambda^{n-1} \xrightarrow{\mathrm{div}} \mathcal{P}_r^-\Lambda^n \to 0$$

which gives the elements of DNA–Falk–Winther '07



σ u p

Other elements:
Cockburn–Gopalakrishnan–Guzmán,
Gopalakrishnan–Guzmán, Stenberg, …

More complexes from complexes

$$
\begin{tikzcd}
0 \arrow[r] & V^1 \arrow[r, "d"] & W^2 \\
0 \arrow[r] & \tilde{V}^1 \arrow[r, "\tilde{d}"] \arrow[ur, "S"] & \tilde{W}^2
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
0 \arrow[r] & \Gamma \arrow[r, "D"] & \tilde{W}^2
\end{tikzcd}
$$

where $\Gamma = \{(v, \tau) \in V^1 \times \tilde{V}^1 \mid dv = S\tau\}$, $D(v, \tau) = \tilde{d}\tau$.

$$
\begin{tikzcd}
0 \arrow[r] & V_h^1 \arrow[r, "d"] & V_h^2 \\
0 \arrow[r] & \tilde{V}_h^1 \arrow[r, "\tilde{d}"] \arrow[ur, "\pi_h^2 S"] & \tilde{W}^2
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
0 \arrow[r] & \Gamma_h \arrow[r, "D"] & \tilde{W}^2
\end{tikzcd}
$$

where $\Gamma_h = \{(v, \tau) \in V_h^1 \times \tilde{V}_h^1 \mid dv = \pi_h^2 S\tau\}$.

Find $u_h \in V_h^1, \sigma_h \in \tilde{V}_h^1, \lambda_h \in V_h^2$ s.t.

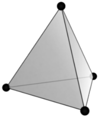
$$
\begin{aligned}
\langle \tilde{d}\sigma_h, \tilde{d}\tau \rangle + \langle \lambda_h, dv - \pi_h S\tau \rangle &= \langle f, v \rangle, \quad && v \in V_h^1,\ \tau \in \tilde{V}_h^1, \\
\langle du_h - \pi_h S\sigma_h, \mu \rangle &= 0, && \mu \in V_h^2.
\end{aligned}
$$

FEEC discretization of the biharmonic

$$\begin{tikzcd}
0 \arrow[r] & \mathring{H}^1(\Omega) \arrow[r, "\operatorname{grad}"] & L^2(\Omega;\mathbb{R}^n) \\
0 \arrow[r] & \mathring{H}^1(\Omega;\mathbb{R}^n) \arrow[r, "\operatorname{grad}"] \arrow[ur, "I"] & L^2_C(\Omega;\mathbb{R}^{n\times n})
\end{tikzcd}
\qquad
\begin{tikzcd}
0 \arrow[r] & \mathcal{P}_r\Lambda^0 \arrow[r, "\operatorname{grad}"] & \mathcal{P}_r^-\Lambda^1 \\
0 \arrow[r] & \mathcal{P}_{r+1}\Lambda^0 \otimes \mathbb{R}^n \arrow[r, "\operatorname{grad}"] \arrow[ur, "\pi_h"] & L^2_C(\Omega;\mathbb{R}^{n\times n})
\end{tikzcd}$$

Find $\quad u_h \in \mathcal{P}_r\Lambda^0, \sigma_h \in \mathcal{P}_{r+1}\Lambda^0, \lambda_h \in \mathcal{P}_r^-\Lambda^1 \quad$ s.t.

$$\begin{aligned}
\langle C \operatorname{grad}\sigma_h, \operatorname{grad}\tau\rangle + \langle \lambda_h, \operatorname{grad} v - \pi_h\tau\rangle &= \langle f, v\rangle, && v \in \mathcal{P}_r\Lambda^0,\ \tau \in \mathcal{P}_{r+1}\Lambda^0, \\
\langle \operatorname{grad} u_h - \pi_h\sigma_h, \mu\rangle &= 0, && \mu \in \mathcal{P}_r^-\Lambda^1.
\end{aligned}$$



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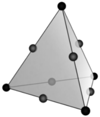

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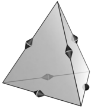

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